$$= \frac{1}{6}(2u_{1,0}^0+2u_{0,1}^0+2u_{0,0}^0)$$

$$= \frac{1}{6}\left(\frac{2}{\sqrt{2}}+\frac{2}{\sqrt{2}}+2\right) = 0.8047$$

$$l = 1, m = 0, u_{1,0}^1 = \frac{1}{6}(u_{2,0}^0+u_{0,0}^0+u_{1,1}^0+u_{1,-1}^0+2u_{1,0}^0)$$

$$= \frac{1}{6}\left(0+1+\frac{1}{2}+\frac{1}{2}+2\,\frac{1}{\sqrt{2}}\right) = 0.5690$$

$$l = 1, m = 1, u_{1,1}^1 = \frac{1}{6}(u_{2,1}^0+u_{0,1}^0+u_{1,2}^0+u_{1,0}^0+2u_{1,1}^0)$$

$$= \frac{1}{6}\left(0+\frac{1}{\sqrt{2}}+0+\frac{1}{\sqrt{2}}+2\,\frac{1}{2}\right) = 0.4024$$

$$n = 1, u_{l,m}^2 = \frac{1}{6}(u_{l+1,m}^1+u_{l-1,m}^1+u_{l,m+1}^1+u_{l,m-1}^1+2u_{l,m}^1)$$

$$l = 0, m = 0, u_{0,0}^2 = \frac{1}{6}(u_{1,0}^1+u_{-1,0}^1+u_{0,1}^1+u_{0,-1}^1+2u_{0,0}^1)$$

$$= \frac{1}{6}(4u_{1,0}^1+2u_{0,0}^1)$$

$$= \frac{1}{6}(4\times 0.5690+2\times 0.8047) = 0.6476$$

$$l = 1, m = 0, u_{1,0}^2 = \frac{1}{6}(u_{2,0}^1+u_{0,0}^1+u_{1,1}^1+u_{1,-1}^1+2u_{1,0}^1)$$

$$= \frac{1}{6}(0+0.8047+2\times 0.4024+2\times 0.5690)$$

$$= 0.4579$$

$$l = 1, m = 1, u_{1,1}^2 = \frac{1}{6}(u_{2,1}^1+u_{0,1}^1+u_{1,2}^1+u_{1,0}^1+2u_{1,1}^1)$$

$$= \frac{1}{6}(0+2\times 0.5690+0+2\times 0.4024)$$

$$= 0.3238$$

5.6.4 Implicit difference schemes

The implicit difference schemes can be derived from the equation

$$F(\nabla_t)u_{l,m}^{n+1} = r[(1+\sigma\delta_x^2)^{-1}\delta_x^2+(1+\sigma\delta_y^2)^{-1}\delta_y^2]\,u_{l,m}^{n+1} \tag{5.171}$$

where $F(\nabla_t)$ is an approximant to $[-\log(1-\nabla_t)]$ and σ is arbitrary. The difference formula

$$[(1-\gamma_1\nabla_t)^{-1}\nabla_t]u_{l,m}^{n+1} = r[(1+\sigma\delta_x^2)^{-1}\delta_x^2+(1+\sigma\delta_y^2)^{-1}\delta_y^2]\,u_{l,m}^{n+1} \tag{5.172}$$

has order of accuracy $(k+h^2)$ for arbitrary γ_1 and σ.

The order of the difference scheme (5.183) is (k^2+h^2) and this order increases to (k^2+h^4) for γ_1, γ_2 arbitrary and $\sigma = 1/12$. Furthermore, from (5.84) it is obvious that the stability (unconditional) of (5.183) depends on the conditions

$$1-2\gamma_1+4\gamma_2 \geqslant 0,\ \gamma_1 \leqslant 1 \text{ and } \sigma < \frac{1}{4} \tag{5.184}$$

The region is shown in Figure 5.5.

Equation (5.183) can be arranged as

$$\begin{aligned}[1+(\sigma-rw_{11})(\delta_x^2+\delta_y^2)+(\sigma^2-2r\sigma w_{11})\delta_x^2\delta_y^2]u_{l,m}^{n+1} \\ = [w_{12}+(\sigma w_{12}-rw_{13})(\delta_x^2+\delta_y^2)+(\sigma^2 w_{12}-2r\sigma w_{13})\delta_x^2\delta_y^2]u_{l,m}^{n} \\ -[w_{14}+(\sigma w_{14}-rw_{15})(\delta_x^2+\delta_y^2)+(\sigma^2 w_{14}-2r\sigma w_{15})\delta_x^2\delta_y^2]u_{l,m}^{n-1}\end{aligned} \tag{5.185}$$

where

$$w_{11} = 2c(1-\gamma_1+\gamma_2),\ w_{12} = 4c(1-\gamma_1),$$
$$w_{13} = 2c(-\gamma_1+2\gamma_2),\ w_{14} = c(1-2\gamma_1),$$
$$w_{15} = 2c\gamma_2, \qquad c = \frac{1}{3-2\gamma_1}$$

In order to factorize the operator on the left-hand side of formula (5.185) the terms $r^2w_{11}^2\delta_x^2\delta_y^2u_{l,m}^{n+1}$ and $r^2w_{11}^2\delta_x^2\delta_y^2(\alpha^*u_{l,m}^{n}+\beta^*u_{l,m}^{n-1})$ are added to left and right sides of (5.185) respectively, leading to

$$\begin{aligned}[1+(\sigma-rw_{11})\delta_x^2][1+(\sigma-rw_{11})\delta_y^2]u_{l,m}^{n+1} = [w_{12}+(\sigma w_{12}-rw_{13})(\delta_x^2+\delta_y^2) \\ +(\sigma^2w_{12}-2r\sigma w_{13}+\alpha^*r^2w_{11}^2)\delta_x^2\delta_y^2]u_{l,m}^{n}-[w_{14}+(\sigma w_{14}-rw_{15})(\delta_x^2+\delta_y^2) \\ +(\sigma^2w_{14}-2rw_{15}-\beta^*r^2w_{11}^2)\delta_x^2\delta_y^2]u_{l,m}^{n-1}\end{aligned} \tag{5.186}$$

where α^* and β^* are arbitrary.

The terms added in (5.186) do not alter the order of accuracy if $\alpha^*+\beta^*=1$, but can have a decided influence on the stability of the formula. In addition to (5.184), the conditions for the scheme to be stable (unconditional) are $\alpha^*+\beta^* = 1$, $\beta^*+1 \geqslant 0$ and $1+\alpha^*-\beta^* \geqslant 0$. A set of values satisfying these conditions are $\alpha^* = 2$, $\beta^* = -1$. The values of the parameters γ_1 and γ_2 can be chosen with the help of Figure 5.5 satisfying the stability conditions.

The multilevel schemes both explicit and implicit for (5.152) can be obtained in a similar manner.

The methods (5.180), (5.181), (5.182) and (5.186) are unconditionally stable. However, the use of these methods requires the solution of a large number of simultaneous linear, algebraic equations at each time step. The iterative methods are usually utilized to obtain the solution. Regardless of the iterative method used, the number of iterations required to achieve a modest numerical accuracy may become large, particularly for large time increments and small space mesh sizes.

We now introduce the methods which are unconditionally stable and in application require much less computation efforts than the above methods.

5.6.3 Alternating Direction Implicit (ADI) methods

The methods are two step methods involving the solution of tridiagonal sets of equations along lines parallel to the x and y axis at the first and second steps respectively. In *Peaceman-Rachford ADI* method, the first step in advancing from t_n to $t_n+k/2$, the implicit differences are used for $\partial^2u/\partial x^2$ and explicit differences are used for $\partial^2u/\partial y^2$. In the second step in advancing from $t_n+k/2$ to t_{n+1} a reversed procedure is used. Accordingly the difference approximation to Equation (5.152) can be expressed as

$$\text{(i)}\quad \frac{u^{n+1/2}_{l,m}-u^n_{l,m}}{k/2} = h^{-2}\delta_x^2 u^{n+1/2}_{l,m} + h^{-2}\delta_y^2 u^n_{l,m}$$

$$\text{(ii)}\quad \frac{u^{n+1}_{l,m}-u^{n+1/2}_{l,m}}{k/2} = h^{-2}\delta_x^2 u^{n+1/2}_{l,m} + h^{-2}\delta_y^2 u^{n+1}_{l,m} \tag{5.187}$$

which may also be written as

$$\text{(i)}\quad \left[1-\frac{r}{2}\delta_x^2\right]u^{n+1/2}_{l,m} = \left[1+\frac{r}{2}\delta_y^2\right]u^n_{l,m}$$

$$\text{(ii)}\quad \left[1-\frac{r}{2}\delta_y^2\right]u^{n+1}_{l,m} = \left[1+\frac{r}{2}\delta_x^2\right]u^{n+1/2}_{l,m} \tag{5.188}$$

The intermediate value $u^{n+1/2}_{l,m}$ can easily be eliminated to obtain an equation relating the solution at t_{n+1} to that at t_n. Subtracting Equations (5.187ii) from (5.187i) and simplifying, we get

$$u^{n+1/2}_{l,m} = \frac{1}{2}(u^{n+1}_{l,m}+u^n_{l,m}) - \frac{r}{4}\delta_y^2(u^{n+1}_{l,m}-u^n_{l,m}) \tag{5.189}$$

If the function value $u^{n+1/2}_{l,m}$ from (5.189) is substituted into (5.187i), it follows that

$$u^{n+1}_{l,m}-u^n_{l,m} = \frac{r}{2}(\delta_x^2+\delta_y^2)(u^{n+1}_{l,m}+u^n_{l,m}) - \frac{r^2}{4}\delta_x^2\delta_y^2(u^{n+1}_{l,m}-u^n_{l,m}) \tag{5.190}$$

which is the same as (5.181)

Another two step alternating direction implicit difference scheme to (5.152) is the *Douglas-Rachford* difference scheme. Here, we first move forward in time in the x-term

$$\frac{u^{*n+1}_{l,m}-u^n_{l,m}}{k} = \frac{1}{2}h^{-2}\delta_x^2(u^{*n+1}_{l,m}+u^n_{l,m}) + h^{-2}\delta_y^2 u^n_{l,m} \tag{5.191}$$

and then correct by moving forward in time in y-term

$$\frac{u^{n+1}_{l,m}-u^n_{l,m}}{k} = \frac{1}{2}h^{-2}\delta_x^2(u^{*n+1}_{l,m}+u^n_{l,m}) + \frac{1}{2}h^{-2}\delta_y^2(u^{n+1}_{l,m}+u^n_{l,m}) \tag{5.192}$$

We have

$$n = 0,\ -u^{1/2}_{l-1,m}+14u^{1/2}_{l,m}-u^{1/2}_{l+1,m} = u^0_{l,m-1}+10u^0_{l,m}+u^0_{l,m+1}$$

$$l = 0,\ m = 0,\ -u^{1/2}_{-1,0}+14u^{1/2}_{0,0}-u^{1/2}_{1,0} = u^0_{0,-1}+10u^0_{0,0}+u^0_{0,1}$$

$$14u^{1/2}_{0,0}-2u^{1/2}_{1,0} = 10+2\frac{1}{\sqrt{2}} = 11.4142$$

$$l = 1,\ m = 0,\ -u^{1/2}_{0,0}+14u^{1/2}_{1,0}-u^{1/2}_{2,0} = u^0_{1,-1}+10u^0_{1,0}+u^0_{1,1}$$

$$-u^{1/2}_{0,0}+14u^{1/2}_{1,0} = 10\frac{1}{\sqrt{2}}+2\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} = 8.0711$$

or

$$\begin{bmatrix} 14 & -2 \\ -1 & 14 \end{bmatrix}\begin{bmatrix} u^{1/2}_{0,0} \\ u^{1/2}_{1,0} \end{bmatrix} = \begin{bmatrix} 11.4142 \\ 8.0711 \end{bmatrix}$$

$$u^{1/2}_{0,0} = 0.9069 \quad u^{1/2}_{1,0} = 0.6413$$

$$l = 1,\ m = 1,\ -u^{1/2}_{1,0}+14u^{1/2}_{1,1}-u^{1/2}_{1,2} = u^0_{1,0}+10u^0_{1,1}+u^0_{1,2}$$

$$14u^{1/2}_{1,1} = \frac{1}{\sqrt{2}}+10\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}+0+0.6413$$

or

$$u^{1/2}_{1,1} = 0.4535$$

$$n = 0,\ -u^1_{l,m-1}+14u^1_{l,m}-u^1_{l,m+1} = u^{1/2}_{l-1,m}+10u^{1/2}_{l,m}+u^{1/2}_{l+1,m}$$

$$l = 0,\ m = 0,\ -u^1_{0,-1}+14u^1_{0,0}-u^1_{0,1} = u^{1/2}_{-1,0}+10u^{1/2}_{0,0}+u^{1/2}_{1,0}$$

$$14u^1_{0,0}-2u^1_{0,1} = 2u^{1/2}_{1,0}+10u^{1/2}_{0,0} = 10.3516$$

$$l = 0,\ m = 1,\ -u^1_{0,0}+14u^1_{0,1}-u^1_{0,2} = u^{1/2}_{-1,1}+10u^{1/2}_{0,1}+u^{1/2}_{1,1}$$

$$-u^1_{0,0}+14u^1_{0,1} = 10u^{1/2}_{0,1}+2u^{1/2}_{1,1} = 7.32$$

or

$$\begin{bmatrix} 14 & -2 \\ -1 & 14 \end{bmatrix}\begin{bmatrix} u^1_{0,0} \\ u^1_{0,1} \end{bmatrix} = \begin{bmatrix} 10.3516 \\ 7.32 \end{bmatrix}$$

$$u^1_{0,0} = 0.8225 \quad u^1_{0,1} = 0.5816$$

$$l = 1,\ m = 1,\ -u^1_{1,0}+14u^1_{1,1}-u^1_{1,2} = u^{1/2}_{0,1}+10u^{1/2}_{1,1}+u^{1/2}_{2,1}$$

$$14u^1_{1,1} = 5.7579$$

or

$$u^1_{1,1} = 0.4113$$

5.7 ADI METHODS FOR EQUATIONS IN TWO SPACE VARIABLES WITH A MIXED DERIVATIVE

We construct the finite difference methods for the solution of the equation

$$\frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + 2B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} \tag{5.201}$$

$$A > 0,\ C > 0,\ B^2 < AC$$

in the region $\overline{\mathcal{R}} = \mathcal{R} \times [0 \leqslant t \leqslant T]$, where $\mathcal{R} = [0 \leqslant x, y \leqslant 1]$.

The initial and boundary conditions to be associated with (5.201) are given by (5.153). The nodal points in $\overline{\mathcal{R}}$ are defined by (5.154).

5.7.1 Two level implicit difference schemes

We write the two level formula as

$$\begin{aligned}
&[1+(\sigma-(1-\gamma_1)rA)\delta_x^2]\,[1+(\sigma-(1-\gamma_1)rC)\delta_y^2]u_{l,m}^{n+1} \\
&= \{[1+(\sigma+\gamma_1 rA)\delta_x^2]\,[1+(\sigma+\gamma_1 rC)\delta_y^2]+(1-2\gamma_1)r^2 A\delta_x^2 C\delta_y^2\}u_{l,m}^n \\
&\quad +2rB\left[C_1(\sigma^{(1)}+\sigma^{(3)})+\left(\frac{1}{2}-C_1\right)(\sigma^{(2)}+\sigma^{(4)})\right]u_{l,m}^n
\end{aligned} \tag{5.202}$$

where σ, γ_1 and C_1 are parameters and

$$\begin{aligned}
\sigma^{(1)}u_{l,m}^n &= u_{l+1,m+1}^n - u_{l,m+1}^n - u_{l+1,m}^n + u_{l,m}^n \\
\sigma^{(2)}u_{l,m}^n &= u_{l,m+1}^n - u_{l-1,m+1}^n - u_{l,m}^n + u_{l-1,m}^n \\
\sigma^{(3)}u_{l,m}^n &= u_{l,m}^n - u_{l-1,m}^n - u_{l,m-1}^n + u_{l-1,m-1}^n \\
\sigma^{(4)}u_{l,m}^n &= u_{l+1,m}^n - u_{l,m}^n - u_{l+1,m-1}^n + u_{l,m-1}^n
\end{aligned} \tag{5.203}$$

Substituting (5.158) into (5.202), the characteristic equation is obtained as

$$\begin{aligned}
&\left[1-4(\sigma-(1-\gamma_1)rA)\sin^2\frac{\theta_1 h}{2}\right]\left[1-4(\sigma-(1-\gamma_1)rC)\sin^2\frac{\theta_2 h}{2}\right]\xi \\
&= \left[1-4(\sigma+\gamma_1 rA)\sin^2\frac{\theta_1 h}{2}\right]\left[1-4(\sigma+\gamma_1 rC)\sin^2\frac{\theta_2 h}{2}\right] \\
&\quad + 16(1-2\gamma_1)r^2 AC\sin^2\frac{\theta_1 h}{2}\sin^2\frac{\theta_2 h}{2} \\
&\quad +8\,rB\left[(4C_1-1)\sin^2\frac{\theta_1 h}{2}\sin^2\frac{\theta_2 h}{2}\right. \\
&\qquad \left. -\sin\frac{\theta_1 h}{2}\cos\frac{\theta_1 h}{2}\sin\frac{\theta_2 h}{2}\cos\frac{\theta_2 h}{2}\right]
\end{aligned} \tag{5.204}$$

The values $\gamma_1 = 1/2$, $\sigma = 1/6$ and $C_1 = 1/4$ give the optimal formula

$$\left[1+\left(\frac{1}{6}-\frac{1}{2}rA\right)\delta_x^2\right]\left[1+\left(\frac{1}{6}-\frac{1}{2}rC\right)\delta_y^2\right]u_{l,m}^{n+1}$$
$$=\left[1+\left(\frac{1}{6}+\frac{1}{2}rA\right)\delta_x^2\right]\left[1+\left(\frac{1}{6}+\frac{1}{2}rC\right)\delta_y^2\right]u_{l,m}^{n}$$
$$+\frac{1}{2}rB[\sigma^{(1)}+\sigma^{(2)}+\sigma^{(3)}+\sigma^{(4)}]\,u_{l,m}^{n} \tag{5.215}$$

Formula (5.202) in split form becomes

(i) $[1+(\sigma-(1-\gamma_1)rA)\delta_x^2]\,u_{l,m}^{*n+1} = [1+(\sigma+\gamma_1 rA)\delta_x^2+rC\delta_y^2$
$+\sigma r(A\delta_x^2\delta_y^2+\delta_x^2C\delta_y^2+2rB(C_1(\sigma^{(1)}+\sigma^{(3)})+(1/2-C_1)(\sigma^{(2)}+\sigma^{(4)}))]u_{l,m}^{n}$

(ii) $[1+(\sigma-(1-\gamma_1)rC)\delta_y^2]u_{l,m}^{n+1} = u_{l,m}^{*n+1}+[(\sigma-(1-\gamma_1)rC)\delta_y^2]u_{l,m}^{n}$ (5.216)

Using (5.153), we obtain the intermediate boundary values from (5.216 ii) and write

$$u_{l,m}^{*n+1} = [1+(\sigma-(1-\gamma_1)rC)\delta_y^2]g_{l,m}^{n+1}$$
$$-[\sigma-(1-\gamma_1)rC]\delta_y^2 g_{l,m}^{n},\ (l,m)\in\partial_x\mathcal{R}_h \tag{5.217}$$

5.7.5 Three level methods

We write the three level difference approximation to (5.201) as

$$[1+(\sigma-p_1rA)\delta_x^2]\,[1+(\sigma-p_1rC)\delta_y^2]u_{l,m}^{n+1}$$
$$=\frac{4(1-\gamma_1)}{3-2\gamma_1}[1+(\sigma+p_2rA)\delta_x^2]\,[1+(\sigma+p_2rC)\delta_y^2]u_{l,m}^{n}$$
$$+\frac{2\gamma_1-1}{3-2\gamma_1}[1+(\sigma+p_3rA)\delta_x^2]\,[1+(\sigma+p_3rC)\delta_y^2]u_{l,m}^{n-1}$$
$$+\left[2p_1^2-\frac{4(1-\gamma_1)}{3-2\gamma_1}p_2^2\right]r^2A\delta_x^2C\delta_y^2u_{l,m}^{n}$$
$$-\left[p_1^2+\frac{2\gamma_1-1}{3-2\gamma_1}p_3^2\right]r^2A\delta_x^2C\delta_y^2\,u_{l,m}^{n-1}$$
$$+\frac{4rB}{3-2\gamma_1}\left[C_1(\sigma^{(1)}+\sigma^{(3)})+\left(\frac{1}{2}-C_1\right)(\sigma^{(2)}+\sigma^{(4)})\right]$$
$$[(2-\gamma_1)u_{l,m}^{n}-(1-\gamma_1)\,u_{l,m}^{n-1}] \tag{5.218}$$

where $\quad p_1=\dfrac{2(1-\gamma_1+\gamma_2)}{3-2\gamma_1},\ p_2=\dfrac{\gamma_1-2\gamma_2}{2(1-\gamma_1)}$

and $\quad p_3=\dfrac{2\gamma_2}{2\gamma_1-1}$

The difference scheme (5.218) has order of accuracy (k^2+h^2) for γ_1, γ_2, σ and C_1 being arbitrary. The stability of the formula can again be discussed by the von-Neumann method. To show that the roots of the characteristic equation governing the three level formula (5.218) lie inside the unit circle, we make the transformation $\xi = (1+z)(1-z)^{-1}$ and apply the Routh-Hurwitz criterion. The transformed equation reduces to

$$\nu_0 z^2+\nu_1 z+\nu_2 = 0 \tag{5.219}$$

where
$$\nu_2 = \frac{8r}{3-2\gamma_1}\Big\{\Big(\sqrt{A}\sin\frac{\theta_1 h}{2}\cos\frac{\theta_2 h}{2}+\frac{B}{\sqrt{A}}\cos\frac{\theta_1 h}{2}\sin\frac{\theta_2 h}{2}\Big)^2$$
$$+[(1-4\sigma)(A+C)-2B(4C_1-1)]\sin^2\frac{\theta_1 h}{2}\sin^2\frac{\theta_2 h}{2}$$
$$+\frac{1}{A}(AC-B^2)\cos^2\frac{\theta_1 h}{2}\sin^2\frac{\theta_2 h}{2}\Big\},$$

$$\nu_1 = \frac{4}{3-2\gamma_1}\Big(1-4\sigma\sin^2\frac{\theta_1 h}{2}\Big)\Big(1-4\sigma\sin^2\frac{\theta_2 h}{2}\Big)+2(1-\gamma_1)\nu_2$$

and
$$\nu_0 = \Big[2r\sqrt{AC}(3-2\gamma_1)\sin\frac{\theta_1 h}{2}\sin\frac{\theta_2 h}{2}$$
$$-\frac{B}{\sqrt{AC}(3-2\gamma_1)}\cos\frac{\theta_1 h}{2}\cos\frac{\theta_2 \gamma}{2}\Big]^2$$
$$+\Big[\frac{16(1-\gamma_1+\gamma_2)^2-(3-2\gamma_1)^4}{(3-2\gamma_1)^2}\Big]4r^2 AC\sin^2\frac{\theta_1 h}{2}\sin^2\frac{\theta_2 h}{2}$$
$$+\frac{2r(1-2\gamma_1+4\gamma_2)}{3-2\gamma_1}\Big(A\sin^2\frac{\theta_1 h}{2}\cos^2\frac{\theta_2 h}{2}+C\sin^2\frac{\theta_2 h}{2}\cos^2\frac{\theta_1 h}{2}\Big)$$
$$+2r\Big[\frac{(1-4\sigma)(1-2\gamma_1+4\gamma_2)}{3-2\gamma_1}(A+C)+2B(4C_1-1)\Big]\sin^2\frac{\theta_1 h}{2}\sin^2\frac{\theta_2 h}{2}$$
$$+\frac{1}{(3-2\gamma_1)^2}\Big[\Big(1-\frac{B^2}{AC}\Big)+2(1-\gamma_1)(3-2\gamma_1)-1\Big]\cos^2\frac{\theta_1 h}{2}\cos^2\frac{\theta_2 h}{2}$$
$$+\frac{2(1-\gamma_1)}{3-2\gamma_1}\Big[(1-4\sigma)\Big(\sin^2\frac{\theta_1 h}{2}\cos^2\frac{\theta_2 h}{2}+\cos^2\frac{\theta_1 h}{2}\sin^2\frac{\theta_2 h}{2}\Big)$$
$$+(1-4\sigma)^2\sin^2\frac{\theta_1 h}{2}\sin^2\frac{\theta_2 h}{2}\Big]$$

Hence the three level scheme is stable if

$$\gamma_1 \leqslant 1,\ 1-2\gamma_1+4\gamma_2 \geqslant 0,\ 2(3-5\gamma_1+2\gamma_1^2)-1 \geqslant 0$$
$$16(1-\gamma_1+\gamma_2)^2-(3-2\gamma_1)^4 \geqslant 0 \tag{5.220}$$

and
$$\sigma < \frac{1}{4},\ |4C_1-1| \leqslant \min\Big[1-4\sigma, \frac{(1-4\sigma)(1-2\gamma_1+4\gamma_2)}{3-2\gamma_1}\Big] \tag{5.221}$$

or
$$\sigma = \frac{1}{4},\ B > 0,\ C_1 \leqslant \frac{1}{4};\ B < 0,\ C_1 \geqslant \frac{1}{4} \tag{5.222}$$

TABLE 5.6 MAXIMUM ABSOLUTE ERRORS

$A = 1,\ B = \frac{1}{2},\ C = 2;\ \gamma_1 = \frac{1}{2},\ C_1 = \frac{1}{4}$; TWO LEVELS

r	Time Steps	σ: $-\frac{1}{3}$	$-\frac{1}{4}$*	0	$\frac{1}{12}$*	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$
0.5	10	0.0046	0.0030	0.0018	0.0035	0.0043	0.0051	0.0068
	20	0.0009	0.0006	0.0003	0.0006	0.0008	0.0009	0.0012
1.0	5	0.0000	0.0016	0.0066	0.0083	0.0091	0.0099	0.0116
	10	0.0000	0.0003	0.0012	0.0015	0.0016	0.0018	0.0021
5.0	1	0.1050	0.1095	0.1237	0.1287	0.1312	0.1338	0.1390
	2	0.0147	0.0159	0.0202	0.0218	0.0227	0.0236	0.0254

**Mckee-Mitchell's* method

TABLE 5.7 MAXIMUM ABSOLUTE ERRORS

$A = \frac{1}{36},\ B = \frac{1}{72},\ C = \frac{1}{18}$, TWO LEVELS $\gamma_1 = \frac{1}{2},\ C_1 = \frac{1}{4}$; THREE LEVELS

$$\gamma_1 = \frac{5-\sqrt{5}}{4},\quad \gamma_2 = \frac{1}{4}\gamma_1$$

(all digits are to be multiplied by 10^{-3})

r	Time Steps	Levels	σ: $-\frac{1}{4}$	0	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$
0.5	10	2	2.14	0.71	0.18	0.09	0.39	1.13
		3	2.00	0.68	0.20	0.05	0.31	1.01
	20	2	4.01	1.33	0.34	0.16	0.68	1.85
		3	3.90	1.34	0.39	0.09	0.59	1.72
1.0	5	2	2.11	0.67	0.15	0.13	0.43	1.17
		3	1.82	0.62	0.18	0.05	0.29	0.94
	10	2	3.95	1.26	0.28	0.23	0.75	1.92
		3	3.74	1.28	0.37	0.09	0.57	1.67
5.0	1	2	1.86	0.39	0.15	0.44	0.77	1.61
	2	2	3.47	0.73	0.27	0.79	1.32	2.54
		3	2.49	0.83	0.22	0.09	0.41	1.25

5.8 ADI METHODS FOR EQUATIONS IN THREE SPACE VARIABLES WITH CONSTANT COEFFICIENTS

Consider the heat flow equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \tag{5.227}$$

in the region $\overline{\mathcal{R}} = \mathcal{R} \times [0 < t \leqslant T]$

where $\mathcal{R} = \{(x, y, z); 0 < x, y, z < 1\}$

with the initial and boundary conditions

$$u(x, y, z, 0) = f(x, y, z), (x, y, z) \in \mathcal{R}$$

$$u(x, y, z, t) = g(x, y, z, t), (x, y, z, t) \in \partial\mathcal{R} \times [0 \leqslant t \leqslant T] \quad (5.228)$$

whe $\partial\mathcal{R}$ is the boundary of $\mathcal{R}$. The mesh sizes are h, k in the space and time coordinates respectively, where $Mh = 1$, $Nk = T$, and so the nodal points are given by

$$\left.\begin{array}{l} x_{l_1} = l_1 h \\ y_{l_2} = l_2 h \\ z_{l_3} = l_3 h \end{array}\right| (l_1, l_2, l_3 = 0, 1, 2, \ldots, M)$$

$$t_n = nk, \ (n = 0, 1, 2, \ldots, N)$$

$$\mathcal{R}_h = \{(l_1, l_2, l_3); l_1, l_2, l_3 = 1, 2, \ldots, M-1\}$$

$$\partial_x\mathcal{R}_h = \{(l_1, l_2, l_3); l_1 = 0, M; l_2, l_3 = 1, 2, \ldots, M-1\}$$

$$\partial_y\mathcal{R}_h = \{(l_1, l_2, l_3); l_2 = 0, M; l_1, l_3 = 1, 2, \ldots, M-1\}$$

$$\partial_z\mathcal{R}_h = \{(l_1, l_2, l_3); l_3 = 0, M; l_1, l_2 = 1, 2, \ldots, M-1\}$$

$$\partial_{yz}\mathcal{R}_h = \{(l_1, l_2, l_3); l_2 = 0, M; l_3 = 0, M; l_1 = 1, 2, \ldots, M-1\}$$

$$\partial_{xz}\mathcal{R}_h = \{(l_1, l_2, l_3); l_1 = 0, M; l_3 = 0, M; l_2 = 1, 2, \ldots, M-1\}$$

$$\partial_{xy}\mathcal{R}_h = \{(l_1, l_2, l_3); l_1 = 0, M; l_2 = 0, M, l_3 = 1, 2, \ldots, M-1\}$$

$$\partial_{xyz}\mathcal{R}_h = \{(l_1, l_2, l_3); l_1 = 0, M; l_2 = 0, M; l_3 = 0, M\} \quad (5.229)$$

with $\partial\mathcal{R}_h = \sum_x \partial\mathcal{R}_h + \sum_{xy} \partial\mathcal{R}_h + \sum_{xyz} \partial\mathcal{R}_h$

Let $u^n_{l_1, l_2, l_3}$ denote the approximate value of $u(x_{l_1}, y_{l_2}, z_{l_3}, t_n)$ and be written as u^n.

If we take first order approximations to $G(\nabla_t)$ and $F(\nabla_t)$ then the two level explicit and implicit difference schemes of order of accuracy $(k+h^2)$ for (5.227) can be obtained as

$$\nabla_t u^{n+1} = r(\delta_x^2+\delta_y^2+\delta_z^2)u^n \quad (5.230)$$

and

$$\nabla_t u^{n+1} = r(\delta_x^2+\delta_y^2+\delta_z^2)u^{n+1} \quad (5.231)$$

The *Crank-Nicolson* two level difference scheme for (5.227) is obtained if a second order rational approximation to $F(\nabla_t)$ is used in (5.31). The difference scheme of $0(k^2+h^2)$ is given by

$$\left(1-\frac{1}{2}\nabla_t\right)^{-1}\nabla_t u^{n+1} = r(\delta_x^2+\delta_y^2+\delta_z^2)\, u^{n+1}$$

or

$$u^{n+1} = u^n + \frac{r}{2}(\delta_x^2+\delta_y^2+\delta_z^2)(u^{n+1}+u^n) \quad (5.232)$$

which may be written as

$$\left(1-\frac{r}{2}\delta_x^2\right)\left(1-\frac{r}{2}\delta_y^2\right)\left(1-\frac{r}{2}\delta_z^2\right)u^{n+1}$$
$$=\left(1+\frac{r}{2}\delta_x^2\right)\left(1+\frac{r}{2}\delta_y^2\right)\left(1+\frac{r}{2}\delta_z^2\right)u^n \qquad (5.233)$$

If we operate on both sides of (5.232) by $(1+\delta_x^2/12)(1+\delta_y^2/12)(1+\delta_z^2/12)$ and group terms correct to $0(k^2+h^2)$, we obtain the *Mitchell-Fairweather formula*

$$\left[1-\frac{1}{2}\left(r-\frac{1}{6}\right)\delta_x^2\right]\left[1-\frac{1}{2}\left(r-\frac{1}{6}\right)\delta_y^2\right]\left[1-\frac{1}{2}\left(r-\frac{1}{6}\right)\delta_z^2\right]u^{n+1}$$
$$=\left[1+\frac{1}{2}\left(r+\frac{1}{6}\right)\delta_x^2\right]\left[1+\frac{1}{2}\left(r+\frac{1}{6}\right)\delta_y^2\right]\left[1+\frac{1}{2}\left(r+\frac{1}{6}\right)\delta_z^2\right]u^n$$
(5.234)

The stability requirement of the difference scheme (5.230) is $0 < r \leqslant 1/6$. The difference schemes (5.231), (5.233) and (5.234) are unconditionally stable. The use of implicit schemes requires the solution of simultaneous equations at every point in the three-dimensional region for each time step. Although for the solution of these there are some very effective methods, this is nevertheless a formidable problem. We now introduce ADI methods which are unconditionally stable but with a greatly reduced amount of computation required for the solution at each time step.

The ADI methods for three space dimensions are the three step methods. The *Douglas-Rachford* ADI method for (5.227) is

(i) $$\frac{u^{*n+1}-u^n}{k} = \frac{1}{h^2}(\delta_x^2 u^{*n+1}+\delta_y^2 u^n+\delta_z^2 u^n)$$

(ii) $$\frac{u^{**n+1}-u^{*n+1}}{k} = \frac{1}{h^2}\delta_y^2 u^{**n+1}-\frac{1}{h^2}\delta_y^2 u^n$$

(iii) $$\frac{u^{n+1}-u^{**n+1}}{k} = \frac{1}{h^2}\delta_z^2 u^{n+1}-\frac{1}{h^2}\delta_z^2 u^n \qquad (5.235)$$

where u^{*n+1} and u^{**n+1} are intermediate values.

Eliminating the intermediate values in (5.235), we obtain

$$u^{n+1} = u^n+r(\delta_x^2+\delta_y^2+\delta_z^2)u^{n+1}-r^2(\delta_x^2\delta_y^2+\delta_y^2\delta_z^2+\delta_z^2\delta_x^2)(u^{n+1}-u^n)$$
$$+r^3\delta_x^2\delta_y^2\delta_z^2(u^{n+1}-u^n) \qquad (5.236)$$

It can be verified that the difference scheme (5.236) has the order of accuracy $(k+h^2)$.

Alternatively, (5.235) may be expressed as

$$\text{(i)}\quad \frac{u^{*n+1}-u^n}{k} = \frac{1}{h^2}\left(\delta_x^2 u^{*n+1}+\delta_y^2 u^n+\delta_z^2 u^n\right)$$

$$\text{(ii)}\quad \frac{u^{**n+1}-u^n}{k} = \frac{1}{h^2}\left(\delta_x^2 u^{*n+1}+\delta_y^2 u^{**n+1}+\delta_z^2 u^n\right)$$

$$\text{(iii)}\quad \frac{u^{n+1}-u^n}{k} = \frac{1}{h^2}\left(\delta_x^2 u^{*n+1}+\delta_y^2 u^{**n+1}+\delta_z^2 u^{n+1}\right) \tag{5.237}$$

Equation (5.237 i) is the same as (5.235 i); the difference between Equations (5.237 ii) and (5.237 i) is equation (5.235 ii); the difference between Equations (5.237 iii) and (5.237 ii) is Equation (5.235 iii).

The *Douglas* ADI method which gives rise to an unconditionally stable difference scheme but which has the accuracy of the Crank-Nicolson scheme (5.232) or (5.233) is

$$\text{(i)}\quad \frac{u^{*n+1}-u^n}{h} = \frac{1}{h^2}\left[\frac{1}{2}\delta_x^2(u^{*n+1}+u^n)+\delta_y^2 u^n+\delta_z^2 u^n\right]$$

$$\text{(ii)}\quad \frac{u^{**n+1}-u^n}{k} = \frac{1}{h^2}\left[\frac{1}{2}\delta_x^2(u^{*n+1}+u^n)+\frac{1}{2}\delta_y^2(u^{**n+1}+u^n)+\delta_z^2 u^n\right]$$

$$\text{(iii)}\quad \frac{u^{n+1}-u^n}{k} = \frac{1}{h^2}\left[\frac{1}{2}\delta_x^2(u^{*n+1}+u^n)+\frac{1}{2}\delta_y^2(u^{**n+1}+u^n)+\frac{1}{2}\delta_z^2(u^{n+1}+u^n)\right] \tag{5.238}$$

The algebraic equations for u^{**n+1} and u^{n+1} can be simplified by subtracting (5.238 i) from (5.238 ii) and (5.238 ii) from (5.238 iii) respectively. After rearrangement, the resulting equations become

$$\text{(i)}\quad \left(\delta_x^2-\frac{2}{r}\right)u^{*n+1} = -\left(\delta_x^2+2\delta_y^2+2\delta_z^2+\frac{2}{r}\right)u^n$$

$$\text{(ii)}\quad \left(\delta_y^2-\frac{2}{r}\right)u^{**n+1} = \delta_y^2 u^n-\frac{2}{r}u^{*n+1}$$

$$\text{(iii)}\quad \left(\delta_z^2-\frac{2}{r}\right)u^{n+1} = \delta_x^2 u^n-\frac{2}{r}u^{**n+1} \tag{5.239}$$

The intermediate solutions u^{*n+1} and u^{**n+1} can be eliminated, using the last two relations of (5.239). The resulting difference equation is

$$u^{n+1} = u^n+\frac{1}{2}r(\delta_x^2+\delta_y^2+\delta_z^2)(u^{n+1}+u^n)-\frac{1}{4}r^2(\delta_x^2\delta_y^2+\delta_y^2\delta_z^2+\delta_z^2\delta_x^2)(u^{n+1}-u^n)$$
$$+\frac{1}{8}r^3\delta_x^2\delta_y^2\delta_z^2(u^{n+1}-u^n) \tag{5.240}$$

The *Brain ADI* method has also the accuracy of Crank-Nicolson method. This method has the following steps:

$$\text{(i)}\quad \frac{u^{*n+1/2}-u^n}{k/2} = \frac{1}{h^2}\left(\delta_x^2 u^{*n+1/2}+\delta_y^2 u^n+\delta_z^2 u^n\right)$$

Similarly, we have

$$Z(x_{m+1/2}, t) = \frac{u(x_{m+1}, t) - u(x_m, t)}{\int_{x_m}^{x_{m+1}} \frac{dx}{p(x, t)}} - \frac{1}{\int_{x_m}^{x_{m+1}} \frac{dx}{p(x, t)}} \int_{x_m}^{x_{m+1}} \frac{dx}{p(x, t)} \int_{x_{m+1/2}}^{x} K(u; s, t)\, ds \tag{5.250}$$

Substituting from (5.249) and (5.250) into (5.248) and integrating between the limits (t_n, t_{n+1}), we obtain the integral identity as follows:

$$\int_{t_n}^{t_{n+1}} \frac{u(x_{m+1}, t) - u(x_m, t)}{\int_{x_m}^{x_{m+1}} \frac{dx}{p(x, t)}}\, dt - \int_{t_n}^{t_{n+1}} \frac{u(x_m, t) - u(x_{m-1}, t)}{\int_{x_{m-1}}^{x_m} \frac{dx}{p(x, t)}}\, dt = \int_{t_n}^{t_{n+1}} dt \int_{x_{m-1/2}}^{x_{m+1/2}} K(u; \lambda, t)\, d\lambda$$

$$+ \int_{t_n}^{t_{n+1}} \frac{dt}{\int_{x_m}^{x_{m+1}} \frac{dx}{p(x, t)}} \int_{x_m}^{x_{m+1}} \frac{dx}{p(x, t)} \int_{x_{m+1/2}}^{x} K(u; s, t)\, ds$$

$$- \int_{t_n}^{t_{n+1}} \frac{dt}{\int_{x_{m-1}}^{x_m} \frac{dx}{p(x, t)}} \int_{x_{m-1}}^{x_m} \frac{dx}{p(x, t)} \int_{x_{m-1/2}}^{x} K(u; \lambda, t)\, d\lambda \tag{5.251}$$

The integrals in (5.251) may be evaluated by the quadrature formulas:

$$\int_{t_n}^{t_{n+1}} f(t)\, dt \approx k[\gamma_1 f(t_n) + (1-\gamma_1) f(t_{n+1})],$$

$$\int_{t_n}^{t_{n+1}} f(t) g'(t)\, dt \approx [\gamma_1 f(t_n) + (1-\gamma_1) f(t_{n+1})][g(t_{n+1}) - g(t_n)]$$

where γ_1, $0 \leqslant \gamma_1 \leqslant 1$ is a parameter,

$$\int_{x_{m-1}}^{x_m} \frac{dx}{\phi(x)} \approx h \frac{1}{\phi(x_{m-1/2})}, \quad \int_{x_m}^{x_{m+1}} \frac{dx}{\phi(x)} \approx h \frac{1}{\phi(x_{m+1/2})}$$

$$\int_{x_{m-1/2}}^{x_{m+1/2}} \phi(x)\, dx \approx h\phi(x_m)$$

$$\int_{x_{m-1}}^{x_m} \frac{dx}{\phi(x)} \int_{x_{m-1/2}}^{x} \psi(s)\, ds \approx 0$$

$$\int_{x_m}^{x_{m+1}} \frac{dx}{\phi(x)} \int_{x_{m+1/2}}^{x} \psi(s)\, ds \approx 0$$

Equation (5.251) can be simplified to

$$\begin{aligned}
&[(1-\gamma_1)c_m^{n+1}+\gamma_1 c_m^n]\,\frac{u_m^{n+1}-u_m^n}{k} \\
&-\frac{(1-\gamma_1)}{h^2}\,[p_{m-1/2}^{n+1}u_{m-1}^{n+1}-(p_{m-1/2}^{n+1}+p_{m+1/2}^{n+1})u_m^{n+1}+p_{m+1/2}^{n+1}u_{m+1}^{n+1}] \\
&-\frac{\gamma_1}{h^2}\,\{p_{m-1/2}^n u_{m-1}^n-(p_{m-1/2}^n+p_{m+1/2}^n)\,u_m^n+p_{m+1/2}^n u_{m+1}^n\} \\
&+(1-\gamma_1)q_m^{n+1}\,u_m^{n+1}+\gamma_1 q_m^n\,u_m^n = (1-\gamma_1)h_m^{n+1}+\gamma_1 h_m^n
\end{aligned} \tag{5.252}$$

where $p_{m\pm1/2}^n = p(x_{m\pm1/2}, t_n)$, $c_m^n = c(x_m, t_n)$,

$q_m^n = q(x_m, t_n)$ and $h_m^n = h(x_m, t_n)$,

which may be rewritten as

$$\begin{aligned}
&[(1-\gamma_1)c_m^{n+1}+\gamma_1 c_m^n]\nabla_t u_m^{n+1}-[(1-\gamma_1)r\delta_x(p_m^{n+1}\delta_x u_m^{n+1}) \\
&+\gamma_1 r\delta_x(p_m^n\delta_x u_m^n)]+(1-\gamma_1)kq_m^{n+1}\,u_m^{n+1}+\gamma_1 kq_m^n\,u_m^n \\
&= (1-\gamma_1)kh_m^{n+1}+\gamma_1 kh_m^n
\end{aligned} \tag{5.253}$$

The truncation error of the difference scheme (5.252) or (5.253) is given by

$$k^{-1}T_m^n = -k\left(\frac{1}{2}-\gamma_1\right)c_m^n\left(\frac{\partial^2 u}{\partial t^2}\right)_m^n + O(k^2+h^2) \tag{5.254}$$

The values $\gamma_1=1$ and $\gamma_1 = 0$ give the explicit and the implicit schemes of order of accuracy $(k+h^2)$ respectively. The value $\gamma_1 = 1/2$ gives the Crank-Nicolson scheme of order (k^2+h^2). The difference schemes for the special

where
$$X_1 = \frac{1}{12} a_1(x_l, y_m, t_{n+1})\delta_x^2 a_1^{-1}(x_l, y_m, t_{n+1})$$
$$X_2 = r a_1(x_l, y_m, t_{n+1})\delta_x^2$$
$$Y_1 = \frac{1}{12} a_2(x_l, y_m, t_{n+1})\delta_y^2 a_2^{-1}(x_l, y_m, t_{n+r})$$
$$Y_2 = r a_2(x_l, y_m, t_{n+1})\delta_y^2 \quad (5.265)$$
is taken as the difference replacement of (5.263).

Expanding (5.264) by Taylor's series about (x_l, y_m, t_{n+1}) and using (5.263), we obtain the leading term of the truncation error as
$$T_l^* = (Y_1X_2 - X_2Y_1)u_{l,m}^{n+1} \quad (5.266)$$
where terms upto $0(h^4)$ have been retained. Therefore, we obtain the difference formula
$$(1+X_1-X_2)(1+Y_1-Y_2)u_{l,m}^{n+1}$$
$$= [(1+X_1)(1+Y_1)+Y_1X_2-X_2Y_1]u_{l,m}^n \quad (5.267)$$
The leading term of the truncation error of (5.267) is given by
$$T_{l,m}^{n+1} = k^2\left[a_1 \frac{\partial^2}{\partial x^2}\left(a_2 \frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{2}\frac{\partial^2 u}{\partial t^2}\right]_{l,m}^{n+1} + kh^4\left[\frac{1}{240}\left(a_1 \frac{\partial^6 u}{\partial x^6} + a_2\frac{\partial^6 u}{\partial y^6}\right)\right.$$
$$- \frac{1}{144}\left\{a_1 \frac{\partial^2}{\partial x^2}\left(a_1^{-1}a_2 \frac{\partial^4}{\partial y^4}\right) + a_2\frac{\partial^2}{\partial y^2}\left(a_2^{-1}a_1 \frac{\partial^4 u}{\partial x^4}\right)\right.$$
$$\left.\left. - a_1\frac{\partial^2}{\partial x^2}\left(a_1^{-1}a_2 \frac{\partial^2}{\partial y^2}\left(a_2^{-1}\frac{\partial u}{\partial t}\right)\right)\right\}\right]_{l,m}^{n+1} + \ldots \quad (5.268)$$
Thus the formula retains its order of accuracy $(k+h^4)$ in two space dimensions.

Using (5.257) as guide, we obtain the *Gourlay-Mitchell* formula of $0(k^2+h^4)$ as
$$[1+X_1^*-X_2^*][1+Y_1^*-Y_2^*]u_{l,m}^{n+1}$$
$$=\{(1+X_1^*+X_2^*)(1+Y_1^*+Y_2^*)-2(X_2^*Y_1^*-Y_1^*X_2^*)\}u_{l,m}^n \quad (5.269)$$
where
$$X_1^* = \frac{1}{12}a_1\delta_x^2 a_1^{-1}, \quad X_2^* = \frac{1}{2}ra_1\delta_x^2,$$
$$Y_1^* = \frac{1}{12}a_2\delta_y^2 a_2^{-1}, \quad Y_2^* = \frac{1}{2}ra_1\delta_y^2$$
with a_1 and a_2 evaluated at $(x_l, y_m, t_{n+1/2})$.

The formula (5.256ii) in two space dimensions can be written as
$$\left[1-\frac{1}{2} r a_{1l,m}^{n+1}\delta_x^2\right]\left[1-\frac{1}{2} r a_{2l,m}^{n+1}\delta_y^2\right]u_{l,m}^{n+1}$$
$$=\left[1+\frac{1}{2} r a_{1l,m}^{n}\delta_x^2\right]\left[1+\frac{1}{2} r a_{2l,m}^{n}\delta_y^2\right]u_{l,m}^{n} \quad (5.270)$$

Similarly, the difference formulas (5.260ii) and (5.260iii) in two space dimensions become

$$\text{(i)}\quad \left(1+X_1-\frac{2}{3}X_2\right)\left(1+Y_1-\frac{2}{3}Y_2\right)u_{l,m}^{n+1}$$
$$= \frac{1}{3}\left[(1+X_1)(1+Y_1)+\frac{4}{9}X_2Y_2\right](4u_{l,m}^{n}-u_{l,m}^{n-1})$$
$$+\frac{2}{3}(Y_1X_2-X_2Y_1)(2u_{l,m}^{n}-u_{l,m}^{n-1})$$

and

$$\text{(ii)}\quad \left(1+X_1-\frac{6}{11}X_2\right)\left(1+Y_1-\frac{6}{11}Y_2\right)u_{l,m}^{n+1}$$
$$= \frac{1}{11}(1+X_1)(1+Y_1)(18u_{l,m}^{n}-9u_{l,m}^{n-1}+2u_{l,m}^{n-2})$$
$$+\frac{36}{121}X_2Y_2\left(\frac{5}{2}u_{l,m}^{n}-2u_{l,m}^{n-1}+\frac{1}{2}u_{l,m}^{n-2}\right)$$
$$+\frac{6}{11}(Y_1X_2-X_2Y_1)(3u_{l,m}^{n}-3u_{l,m}^{n-1}+u_{l,m}^{n-2}) \qquad (5.271)$$

respectively, where X_1, X_2, Y_1 and Y_2 are defined in (5.265). The compact implicit difference schemes (5.262) can be extended easily to (5.263). For example, (5.262 i) becomes

$$\left[1-\frac{r}{2}a_1^{n+1}Q_x^{-1}\delta_x^2\right]\left[1-\frac{r}{2}a_2^{n+1}Q_y^{-1}\delta_y^2\right]u_{l,m}^{n+1}$$
$$=\left[1+\frac{r}{2}a_1^{n}Q_x^{-1}\delta_x^2\right]\left[1+\frac{r}{2}a_2^{n}Q_y^{-1}\delta_y^2\right]u_{l,m}^{n} \qquad (5.272)$$

where a_1^s and a_2^s, $s = n, n+1$ are evaluated at $x = lh$, $y = mh$.

5.9.3 Three space dimensions

Here we consider the equation

$$\frac{\partial u}{\partial t} = a_1\frac{\partial^2 u}{\partial x^2}+a_2\frac{\partial^2 u}{\partial y^2}+a_3\frac{\partial^2 u}{\partial z^2} \qquad (5.273)$$

in the region $\mathcal{R} = [0 \leqslant x, y, z \leqslant 1]\times[0 \leqslant t \leqslant T]$ with the initial and boundary conditions (5.228).

It is simple to extend the formulas (5.267), (5.269) and (5.271) to three space dimensions. The following difference schemes are obtained respectively:

$$(1+X_1-X_2)(1+Y_1-Y_2)(1+Z_1-Z_2)u^{n+1} = [(1+X_1)(1+Y_1)(1+Z_1)$$
$$+Y_1X_2-X_2Y_1+Z_1X_2-X_2Z_1+Z_1Y_2-Y_2Z_1]u^{n} \qquad (5.274)$$

THEOREM (*Widlund*) 5.2. *Let* (5.279) *be a consistent difference approximation to* (5.278). *Let*

(*i*) *the eigenvalues* λ_i, $i = 2, \ldots, v+1$ *of D lie in the interior of the unit circle and if they are of modulus unity they are simple,*

(*ii*) *for every* $\beta \neq 0$ *all eigenvalues of* $\hat{Q}$ *lie in the interior of the unit circle, then the difference scheme is strongly stable.*

To establish the stability of the formulas derived earlier, we follow the above analysis. In *Widlund's* notations we write (5.260 iii) as

$$u_m^{n+1} = \frac{18}{11}u_m^n - \frac{9}{11}u_m^{n-1} + \frac{2}{11}u_m^{n-2} + \sum_{p=-1}^{2} Q_p u_m^{n-p} \tag{5.284}$$

where

$$Q_{-1} = -\left(X_1 - \frac{6}{11}X_2\right)$$

$$Q_0 = \frac{18}{11}X_1$$

$$Q_1 = -\frac{9}{11}X_1$$

$$Q_2 = \frac{2}{11}X_1$$

The principal parts are given by

$$Q_{-1}^{(1)} = -\frac{1}{12}(hD_{+x})(hD_{-x}) + \frac{6}{11}ra_1(hD_{+x})(hD_{-x})$$

$$Q_0^{(1)} = \frac{3}{22}(hD_{+x})(hD_{-x})$$

$$Q_1^{(1)} = -\frac{3}{44}(hD_{+x})(hD_{-x})$$

$$Q_2^{(1)} = \frac{1}{66}(hD_{+x})(hD_{-x})$$

Replacing now $hD_{\pm j}$ by $2i\sin(\beta_1 h/2)\exp(\pm\frac{1}{2}i\beta_1 h)$ we get

$$\hat{Q}_{-1}^{(1)} = \frac{1}{3}\sin^2\frac{\beta_1 h}{2} - \frac{24}{11}ra_1\sin^2\frac{\beta_1 h}{2}$$

$$\hat{Q}_0^{(1)} = -\frac{6}{11}\sin^2\frac{\beta_1 h}{2}$$

$$\hat{Q}_1^{(1)} = \frac{3}{11}\sin^2\frac{\beta_1 h}{3}$$

$$\hat{Q}_2^{(1)} = -\frac{2}{33}\sin^2\frac{\beta_1 h}{2}$$

The matrix **D** in this case is given by

$$\mathbf{D} = \begin{bmatrix} \frac{18}{11} & \frac{9}{11} & \frac{2}{11} \\ 1 & \eta & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{5.285}$$

It is easily verified that the two eigenvalues of D lie inside the unit circle and one on the unit circle. Furthermore,

$$(1-\hat{Q}^{(1)}_{-1})^{-1}\left(\frac{18}{11}+\hat{Q}^{(1)}_{0}\right) = \frac{18s_1}{11s_1+s_2}$$

$$(1-\hat{Q}^{(1)}_{-1})^{-1}\left(-\frac{9}{11}+\hat{Q}^{(1)}_{1}\right) = \frac{-9s_1}{11s_1+s_2}$$

$$(1-\hat{Q}^{(1)}_{-1})^{-1}\left(\frac{2}{11}+\hat{Q}^{(1)}_{2}\right) = \frac{2s_1}{11s_1+s_2}$$

where $s_1 = 3-\sin^2(\beta_1 h/2)$, $s_2 = 72ra_1 \sin^2(\beta_1 h/2)$, then we derive the matrix $\hat{Q}$ to be

$$\hat{Q} = \begin{bmatrix} \frac{18s_1}{11s_1+s_2} & \frac{-9s_1}{11s_1+s_2} & \frac{2s_1}{11s_1+s_2} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{5.286}$$

The characteristic equation of (5.286) is obtained as

$$(11s_1+s_2)\xi^3+18s_1\xi^2+9s_1\xi-2s_1 = 0 \tag{5.287}$$

Using the transformation $\xi = (1+z)/(1-z)$ into (5.287), the characteristic equation takes the form

$$V_0z^3+V_1z^2+V_2z+V_3 = 0$$

where

$$V_0 = 40s_1+s_2, \quad V_1 = 3(12s_1+s_2)$$
$$V_2 = 3(4s_1+s_2), \quad V_3 = s_2$$

and

$$V_1V_2-V_0V_3 = 8(54s_1^2+13s_1s_2+s_2^2) \tag{5.288}$$

All the quantities defined in (5.288) are positive and we have by Routh-Hurwitz criterion that $|\xi_i| < 1$, $1 \leqslant i \leqslant 3$. Hence the difference scheme (5.260) is unconditionally stable.

Next we rewrite the difference scheme (5.271 ii) in two space dimensions as

$$u^{n+1}_{l,m} = \frac{18}{11}u^n_{l,m}-\frac{9}{11}u^{n-1}_{l,m}+\frac{2}{11}u^{n-2}_{l,m}+\sum_{p=-1}^{2} Q_p u^{n-p}_{l,m} \tag{5.289}$$

where

$$Q_{-1} = 1-\left(1+X_1-\frac{6}{11}X_2\right)\left(1+Y_1-\frac{6}{11}Y_2\right)$$

$$Q_0 = \frac{18}{11}[(1+X_1)(1+Y_1)-1]+\frac{90}{121}X_2Y_2+\frac{18}{11}(Y_1X_2-X_2Y_1)$$

$$Q_1 = -\frac{9}{11}[(1+X_1)(1+Y_1)-1]-\frac{72}{121}X_2Y_2-\frac{18}{11}(Y_1X_2-X_2Y_1)$$

$$Q_2 = \frac{2}{11}[(1+X_1)(1+Y_1)-1]+\frac{18}{121}X_2Y_2+\frac{6}{11}(Y_1X_2-X_2Y_1)$$

We find

$$\hat{Q} = \begin{bmatrix} \dfrac{\frac{18}{11}S_1^*+\frac{5}{2}S_3}{S_1^*+S_2+S_3} & \dfrac{-\frac{9}{11}S_1^*-2S_3}{S_1^*+S_2+S_3} & \dfrac{\frac{2}{11}S_1^*+\frac{1}{2}S_3}{S_1^*+S_2+S_3} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{5.290}$$

where

$$S_1^* = \left(1-\frac{1}{3}\sin^2\frac{\beta_1 h}{2}\right)\left(1-\frac{1}{3}\sin^2\frac{\beta_2 h}{2}\right),$$

$$S_2 = \frac{24}{11}r\left[\left(1-\frac{1}{3}\sin^2\frac{\beta_1 h}{2}\right)a_2\sin^2\frac{\beta_2 h}{2}\right.$$
$$\left.+\left(1-\frac{1}{3}\sin^2\frac{\beta_2 h}{2}\right)a_1\sin^2\frac{\beta_1 h}{2}\right]$$

$$S_3 = \frac{576}{121}r^2a_1a_2\sin^2\frac{\beta_1 h}{2}\sin^2\frac{\beta_2 h}{2}$$

The characteristic equation of (5.290) is obtained as

$$11(S_1^*+S_2+S_3)\,\xi^3-\left(18S_1^*+\frac{55}{2}S_3\right)\xi^2$$
$$+(9S_1^*+22S_3)\,\xi-\left(2S_1^*+\frac{11}{2}S_3\right) = 0 \tag{5.291}$$

Substituting $\xi = (1+z)/(1-z)$ in (5.291), we obtain

$$V_0z^3+V_1z^2+V_2z+V_3 = 0$$

where

$$V_0 = 10S_1+S_2^*+3S_3^*,$$
$$V_1 = 9S_1+3S_2^*+S_3^*,$$
$$V_2 = 3S_1+3S_2^*,$$
$$V_3 = S_2^*$$

with

$$S_1 = 4S_1^*,\ 11S_2 = S_2^*,\ 22S_3 = S_3^*$$

Again, all the quantities V_0, V_1, V_2, $V_3 > 0$ and

$$27S_1^2 + 26S_1S_2^* + 3S_1S_3^* + 8S_2^{*2} > 0$$

and we have $|\xi_i| < 1$, $i = 1, 2, 3$. Thus, the difference scheme (5.271ii) is unconditionally stable.

5.9.5 ADI formulas

In order to facilitate computations, thc difference schemes can be rewritten in a split form. The intermediate boundary values on $\partial\mathfrak{R}$ can be derived from the split equations. We write below some split formulas in two dimensions.

The two level formula (5.267) of $0(k+h^2)$ can be written in split form as

$$(1+X_1-X_2)u_{l,m}^{*n+1} = (X_2+Y_1X_2)u_{l,m}^n$$

$$(1+Y_1-Y_2)u_{l,m}^{n+1} = u_{l,m}^{*n+1}+(1+Y_1)u_{l,m}^n \qquad (5.292)$$

The intermediate boundary values are given by

$$g_{l,m}^{*n+1} = (1+Y_1-Y_2)g_{l,m}^{n+1}-(1+Y_1)g_{l,m}^n \qquad (5.293)$$

Formula (5.269) may be expressed as

$$(1+X_1^*-X_2^*)u_{l,m}^{*n+1} = [(1+X_1^*)(1+Y_1^*+Y_2^*) - (X_2^*Y_1^*-Y_1^*X_2^*)]u_{l,m}^n$$

$$(1+Y_1^*-Y_2^*)u_{l,m}^{n+1} = 2u_{lm}^{*n+1}-(1+Y_1^*+Y_2^*)u_{l,m}^n \qquad (5.294)$$

with the intermediate boundary conditions

$$g_{lm}^{*n+1} = \tfrac{1}{2}(1+Y_1^*-Y_2^*)g_{l,m}^{n+1}+\tfrac{1}{2}(1+Y_1^*+Y_2^*)g_{l,m}^n \qquad (5.295)$$

Formula (5.270) can be written as

$$\left(1-\frac{r}{2}a_{1l,m}^{n+1}\,\delta_x^2\right)u_{l,m}^{*n+1} = \left(1+\frac{r}{2}\,a_{1l,m}^n\,\delta_x^2\right)\left(1+\frac{r}{2}\,a_{2l,m}^n\,\delta_y^2\right)u_{l,m}^n$$

$$\left(1-\frac{r}{2}\,a_{2l,m}^{n+1}\delta_y^2\right)u_{l,m}^{n+1} = u_{l,m}^{*n+1} \qquad (5.296)$$

with intermediate boundary conditions

$$g_{l,m}^{*n+1} = \left(1-\frac{r}{2}\,a_{2l,m}^{n+1}\,\delta_y^2\right)g_{l,m}^{n+1} \qquad (5.297)$$

The three level formula(5.271i) of $0(k^2+h^4)$ can be splitted as

$$\left(1+X_1-\frac{2}{3}X_2\right)u_{l,m}^{*n+1} = \frac{2}{9}X_2\left(1+Y_1+\frac{2}{3}\,Y_2\right)(4u_{l,m}^n-u_{l,m}^{n-1}) + \frac{2}{3}(Y_1X_2-X_2Y_1)(2u_{l,m}^n-u_{l,m}^{n-1})$$

$$\left(1+Y_1-\frac{2}{3}\,Y_2\right)u_{l,m}^{n+1} = u_{l,m}^{*n+1}+\frac{1}{3}(1+Y_1)(4u_{l,m}^n-u_{l,m}^{n-1}) \qquad (5.298)$$

TABLE 5.9 MAXIMUM ABSOLUTE ERRORS, EXAMPLE (iii)

Two levels: $\gamma_1 = 1/2, C_1 = 1/4$; *Three levels:* $\gamma_1 = (5-\sqrt{5})/4, \gamma_2 = \gamma_1/2$

(ALL DIGITS ARE TO BE MULTIPLIED BY 10^{-3})

r	*Time*	*Time steps*	*Levels*	σ = $-\frac{1}{4}$*	σ = 0	σ = $\frac{1}{12}$*	σ = $\frac{1}{8}$	σ = $\frac{1}{6}$	σ = $\frac{1}{4}$
0.5	$\frac{1}{20}$	10	2	1.28	0.24	0.24	0.43	0.62	1.04
			3	1.02	0.00	0.37	0.55	0.74	1.14
	$\frac{1}{10}$	20	2	2.02	0.33	0.39	0.69	1.09	1.62
			3	1.69	0.00	0.60	0.91	1.23	1.89
	$\frac{3}{20}$	30	2	2.47	0.39	0.49	0.85	1.25	1.99
			3	2.15	0.00	0.76	1.15	1.54	2.37
1.0	$\frac{1}{20}$	5	2	1.47	0.48	0.18	0.31	0.49	0.89
			3	0.94	0.00	0.34	0.51	0.69	1.05
	$\frac{1}{10}$	10	2	2.29	0.66	0.27	0.52	0.79	1.41
			3	1.64	0.01	0.59	0.89	1.20	1.84
	$\frac{3}{20}$	15	2	2.83	0.76	0.34	0.64	0.99	1.75
			3	2.11	0.01	0.75	1.14	1.52	2.34
5.0	$\frac{1}{20}$	1	2	3.05	2.21	1.91	1.77	1.60	1.39
	$\frac{1}{10}$	2	2	4.49	3.10	2.65	2.44	2.22	1.77
			3	1.02	0.11	0.49	0.69	0.89	1.30
	$\frac{3}{20}$	3	2	5.40	3.57	2.96	2.69	2.41	1.84
			3	1.61	0.18	0.77	1.08	1.39	2.02

*Mckee-Mitchell method.

TABLE 5.10 MEAN DEVIATION FROM MEAN, EXAMPLE (iii)
(ALL DIGITS ARE TO BE MULTIPLIED BY 10^{-3})

r	*Time*	*Time steps*	*Levels*	σ					
				$-\frac{1}{4}$*	0	$\frac{1}{12}$*	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{4}$
	$\frac{1}{20}$	10	2	0.26	0.06	0.06	0.09	0.13	0.21
			3	0.21	0.00	0.07	0.11	0.15	0.23
0.5	$\frac{1}{10}$	20	2	0.42	0.08	0.09	0.15	0.21	0.34
			3	0.36	0.00	0.13	0.19	0.25	0.39
	$\frac{3}{20}$	30	2	0.53	0.09	0.12	0.19	0.27	0.43
			3	0.46	0.00	0.16	0.24	0.32	0.49
	$\frac{1}{20}$	5	2	0.30	0.11	0.09	0.10	0.12	0.19
			3	0.19	0.00	0.07	0.10	0.14	0.21
1.0	$\frac{1}{10}$	10	2	0.48	0.16	0.13	0.15	0.19	0.30
			3	0.34	0.00	0.12	0.18	0.25	0.38
	$\frac{3}{20}$	15	2	0.60	0.18	0.14	0.18	0.24	0.38
			3	0.45	0.00	0.16	0.24	0.32	0.49
	$\frac{1}{20}$	1	2	0.66	0.53	0.50	0.49	0.48	0.48
5.0	$\frac{1}{10}$	2	2	0.96	0.37	0.67	0.65	0.63	0.62
			3	0.21	0.03	0.10	0.14	0.18	0.26
	$\frac{3}{20}$	3	2	1.16	0.82	0.75	0.72	0.70	0.68
			3	0.33	0.04	0.16	0.22	0.28	0.41

To derive implicit difference schemes, we use rational approximation to derivatives in (5.300) and write as

$$[(1+\tau\delta_t^2)^{-1}\delta_t^2+r^2(1+\beta\delta_x^2+\beta_1\delta_x^4)^{-1}\delta_x^4]\,u_m^n=0$$

or

$$[1+\beta\delta_x^2+(\beta_1+\tau r^2)\delta_x^4]\,\delta_t^2u_m^n+r^2\delta_x^4u_m^n=0 \tag{5.310}$$

where β, β_1 and τ are arbitrary parameters. Substituting the relations

$$\delta_t^2u(x_m,t_n)=-r^2D_x^4+\frac{r^4}{12}D_x^8-\frac{r^6}{360}D_x^{12}+\dots$$

$$\delta_x^2\delta_t^2u(x_m,t_n)=-r^2D_x^6-\frac{r^2}{12}D_x^8+\frac{1}{12}\left(r^4-\frac{r^2}{30}\right)D_x^{10}-\dots$$

$$\delta_x^4u(x_m,t_n)=D_x^4+\frac{1}{6}D_x^6+\frac{1}{80}D_x^8+\frac{17}{30240}D_x^{10}+\dots$$

$$\delta_x^4\delta_t^2u(x_m,t_n)=-r^2D_x^8-\frac{r^2}{6}D_x^{10}-\dots$$

where

$$D_x^i=h^i\frac{\partial^i u}{\partial x^i}\bigg|_m^n$$

into (5.310) and simplifying, we get the truncation error

$$\begin{aligned}T_m^n=r^2\left(\frac{1}{6}-\beta\right)h^6\left(\frac{\partial^6u}{\partial x^6}\right)_m^n+\bigg[\left(\frac{1}{12}-\tau\right)r^4\\+\left(\frac{1}{80}-\beta_1-\frac{1}{12}\beta\right)r^2\bigg]h^8\left(\frac{\partial^8u}{\partial x^8}\right)_m^n+\bigg[\frac{1}{6}\left(\frac{1}{2}\beta-\tau\right)r^4\\-\frac{1}{6}\left(\beta_1+\frac{1}{60}\beta\right)r^2+\frac{17}{30240}r^2\bigg]h^{10}\left(\frac{\partial^{10}u}{\partial x^{10}}\right)_m^n+\dots\end{aligned} \tag{5.311}$$

From (5.311) it is obvious that the difference scheme (5.310) is of the order of accuracy (k^2+h^2) for the arbitrary parameters.

Using the von Neumann method, we can show that the difference scheme (5.310) is stable if

$$-1\leqslant 1-\frac{8r^2\sin^4\dfrac{\theta_1h}{2}}{1-4\beta\sin^2\dfrac{\theta_1h}{2}+16(\beta_1+\tau r^2)\sin^4\dfrac{\theta_1h}{2}}\leqslant 1 \tag{5.312}$$

The right side of this inequality is trivially satisfied if $\beta\leqslant 1/4$ and β_1, $\tau>0$.

From the left side inequality, we obtain

$$1-4\beta\sin^2\frac{\theta_1h}{2}+4(4\beta_1+(4\tau-1)r^2)\sin^4\frac{\theta_1h}{2}\geqslant 0$$

Hence, we find that the formula (5.310) is unconditionally stable if

$$\beta_1 = 0, \beta < \frac{1}{4}, \tau \geqslant \frac{1}{4},$$

$$\beta_1 \geqslant \frac{1}{4}\beta^2, \tau \geqslant \frac{1}{4}$$

and conditionally stable if

$$\beta_1 = 0, \tau < \frac{1}{4}, \beta < \frac{1}{4}, 0 < r^2 \leqslant \frac{1-4\beta}{4(1-4\tau)},$$

$$\beta_1 = \frac{1}{4}\beta^2, \tau < \frac{1}{4}, 0 < r^2 \leqslant \frac{(1-2\beta)^2}{4(1-4\tau)}$$

The stability region is shown in Figure 5.10. The parameters β, β_1 and τ may be so chosen that the difference scheme (5.310) is not only accurate but also stable. From Figure 5.10 and Equation (5.311), we find that the values

$$\beta = \frac{1}{6}, \quad \beta_1 = 0 \quad \text{or} \quad \geqslant \frac{1}{4}\beta^2 \quad \text{and} \quad \tau \geqslant \frac{1}{4}$$

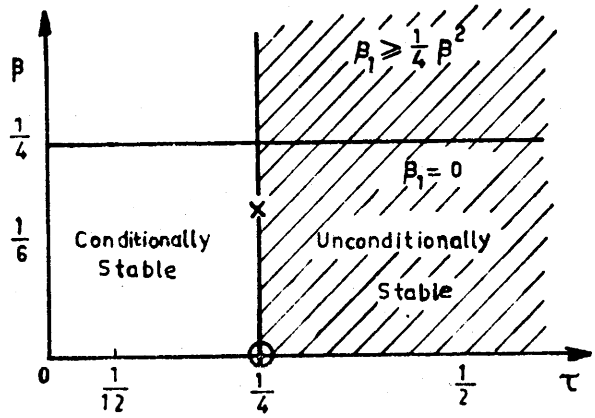

Fig. 5.10 Representation of stability region

give unconditionally stable formulas. In particular, for $\beta = 1/6$, $\tau = 1/4$, $\beta_1 = 0$ or $\geqslant \beta^2/4$, we get unconditionally stable formulas which have the minimum truncation error from the class of formulas with order of accuracy (k^2+h^4). Thus, the optimal formulas are given by

(i) $\left(1+\frac{1}{6}\delta_x^2+\frac{1}{4}r^2\delta_x^4\right)\delta_t^2 u_m^n + r^2\delta_x^4 u_m^n = 0$

(ii) $\left(1+\frac{1}{6}\delta_x^2+\left(\frac{1}{144}+\frac{1}{4}r^2\right)\delta_x^4\right)\delta_t^2 u_m^n + r^2\delta_x^4 u_m^n = 0$ (5.313)

$$m = 1,\ u_1^2 = 2u_1^1 - u_1^0 - \frac{1}{4}(u_{-1}^1 - 4u_0^1 + 6u_1^1 - 4u_2^1 + u_3^1)$$

$$= -\frac{377}{24576} = -0.0153$$

$$m = 2,\ u_2^2 = 2u_2^1 - u_2^0 - \frac{1}{4}(u_0^1 - 4u_1^1 + 6u_2^1 - 4u_3^1 + u_4^1)$$

$$= -\frac{268}{12288} = -0.0218$$

$$m = 3,\ u_3^2 = 2u_3^1 - u_3^0 - \frac{1}{4}(u_1^1 - 4u_2^1 + 6u_3^1 - 4u_4^1 + u_5^1)$$

$$= -\frac{377}{24576} = -0.0153$$

$$n = 2,\ u_m^3 - 2u_m^2 + u_m^1 + \frac{1}{4}(u_{m-2}^2 - 4u_{m-1}^2 + 6u_m^2 - 4u_{m+1}^2 + u_{m+2}^2) = 0$$

$$1 \leqslant m \leqslant 3$$

$$m = 1,\ u_1^3 = 2u_1^2 - u_1^1 - \frac{1}{4}(u_{-1}^2 - 4u_0^2 + 6u_1^2 - 4u_2^2 + u_3^2)$$

$$= -\frac{581}{49152} = -0.0118$$

$$m = 2,\ u_2^3 = 2u_2^2 - u_2^1 - \frac{1}{4}(u_0^2 - 4u_1^2 + 6u_2^2 - 4u_3^2 + u_4^2)$$

$$= -\frac{203}{12288} = -0.0165$$

$$m = 3,\ u_3^3 = 2u_3^2 - u_3^1 - \frac{1}{4}(u_1^2 - 4u_2^2 + 6u_3^2 - 4u_4^2 + u_5^2)$$

$$= -\frac{581}{49152} = -0.0118$$

The Albrecht method, for $r = \frac{1}{2}$ becomes

$$2u_m^{n+1} - 2(u_{m-1}^n + u_{m+1}^n) + (u_{m-2}^{n-1} + u_{m+2}^{n-1}) + 2u_m^{n-1} - 2(u_{m-1}^{n-2} + u_{m+1}^{n-2}) + 2u_m^{n-3} = 0$$

This is a 4-step method. We use the Collatz method to start the computation.

We have

$$n = 3,\ 2u_m^4 - 2(u_{m-1}^3 + u_{m+1}^3) + (u_{m-2}^2 + u_{m+2}^2 + 2u_m^2)$$
$$- 2(u_{m-1}^1 + u_{m+1}^1) + 2u_m^0 = 0 \qquad 1 \leqslant m \leqslant 3$$

$$m = 1,\ u_1^4 = u_0^3 + u_2^3 - \frac{1}{2}(u_{-1}^2 + u_3^2 + 2u_1^2) + u_0^1 + u_2^1 - u_1^0$$

$$= -\frac{189}{24576} = -0.0077$$

$$m = 2,\ u_2^4 = u_1^3 + u_3^3 - \frac{1}{2}(u_0^2 + u_4^2 + 2u_2^2) + u_1^1 + u_3^1 - u_2^0$$

$$= -\frac{273}{24576} = -0.0111$$

$$m = 3,\ u_3^4 = u_2^3 + u_4^3 - \frac{1}{2}(u_1^2 + u_3^2 + 2u_3^2) + u_2^1 + u_4^1 - u_3^0$$

$$= -\frac{189}{24576} = -0.0077$$

The Conte method for $r = \frac{1}{2}$ becomes

$$4(u_m^{n+1} - 2u_m^n + u_m^{n-1}) - (u_{m+1}^{n+1} - 2u_m^{n+1} + u_{m-1}^{n+1})$$
$$+ 2(u_{m+1}^n - 2u_m^n + u_{m-1}^n) - (u_{m+1}^{n-1} - 2u_m^{n-1} + u_{m-1}^{n-1})$$
$$+ u_{m-2}^n - 4u_{m-1}^n + 6u_m^n - 4u_{m+1}^n + u_{m+2}^n = 0 \qquad 1 \leqslant m \leqslant 3$$

We have

$$n = 0,\ -2u_{m+1}^1 + 12u_m^1 - 2u_{m-1}^1 + u_{m-2}^0 - 2u_{m-1}^0 - 6u_m^0 - 2u_{m+1}^0 + u_{m+2}^0 = 0$$
$$1 \leqslant m \leqslant 3$$

$$m = 1,\ -2u_2^1 + 12u_1^1 - 2u_1^0 + u_{-1}^0 - 2u_0^0 - 6u_1^0 - 2u_2^0 + u_3^0 = 0$$

$$6u_1^1 - u_2^1 = -\frac{502}{6144}$$

$$m = 2,\ -2u_3^1 + 12u_2^1 - 2u_1^1 + u_0^0 - 2u_1^0 - 6u_2^0 - 2u_3^0 + u_4^0 = 0$$

$$-u_1^1 + 6u_2^1 - u_3^1 = -\frac{708}{6144}$$

$$m = 3,\ -2u_4^1 + 12u_3^1 - 2u_2^1 + u_1^0 - 2u_2^0 - 6u_3^0 - 2u_4^0 + u_5^0 = 0$$

$$-u_2^1 + 6u_3^1 = -\frac{502}{6144}$$

The system of equations may be written as

$$\begin{bmatrix} 6 & -1 & 0 \\ -1 & 6 & -1 \\ 0 & -1 & 6 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = -\frac{1}{6144} \begin{bmatrix} 502 \\ 708 \\ 502 \end{bmatrix}$$

Solving we obtain

$$u_1^1 = -0.0178, \quad u_2^1 = -0.0251, \quad u_3^1 = -0.0178$$

Using the above difference schemes we find the quantities v_m^n and w_m^n. The function values u_m^n may be obtained from v_m^n and w_m^n using the Numerov method

$$u_{m+1}^n - 2u_m^n + u_{m-1}^n = \frac{h^2}{12}(w_{m+1}^n + 10w_m^n + w_{m-1}^n)$$

with local truncation error of $0(h^6)$ or a high order method

$$u_{m+1}^n - 2u_m^n + u_{m-1}^n = \frac{h^2}{12}(w_{m+1}^n + 10w_m^n + w_{m-1}^n) - \frac{h^2}{240r}\delta_x^2(v_m^n - v_m^{n-1}) \tag{5.327}$$

with local truncation error of $0(h^8)$.

5.10.3 Results from computation

We have solved (5.300) together with the initial conditions

$$u(x, 0) = \frac{x}{12}(2x^2 - x^3 - 1), \frac{\partial u(x, 0)}{\partial t} = 0, 0 \leqslant x \leqslant 1$$

and the boundary conditions

$$u(0, t) = u(1, t) = 0$$

$$\frac{\partial^2 u(0, t)}{\partial x^2} = \frac{\partial^2 u(1, t)}{\partial x^2} = 0, \quad t \geqslant 0$$

The theoretical solution is given by

$$u(x, t) = \sum_{s=0}^{\infty} d_s \sin (2s+1)\pi x \cos (2s+1)^2\pi^2 t$$

where

$$d_s = \frac{8}{(2s+1)^5\pi^5}$$

The initial and boundary conditions on v and w may be derived from the above conditions and are given by

$$\begin{bmatrix} v(x, 0) \\ w(x, 0) \end{bmatrix} = \begin{bmatrix} 0 \\ x - x^2 \end{bmatrix}, \quad 0 \leqslant x \leqslant 1$$

and

$$\begin{bmatrix} v(0, t) \\ w(0, t) \end{bmatrix} = \begin{bmatrix} v(1, t) \\ w(1, t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad t \geqslant 0$$

The difference scheme (5.325) with $\sigma = 1/12$ and different values of γ_1, γ_2 has been used to solve this problem. Formula (5.327) is used to find the function $u(x, t)$ when applying (5.325) and the Numerov method is used otherwise. We choose $h = .1$ and $k = .02$ which gives $r = 2.0$. The computations are carried up to $t = 1$ giving 50 time steps. The error values $|u_m^n - u(x_m, t_n)|$ are tabulated in Table 5.11. It is observed that all points on the line $1-2\gamma_1+4\gamma_2 = 0$ produce accurate results. The values $\gamma_1 = 1/2$ and $\gamma_2 = 0$ give the Fairweather-Gourlay two level difference scheme. It can be seen that there are many values of (γ_1, γ_2) on $1-2\gamma_1+4\gamma_2 = 0$ which give better results than the two level scheme.

TABLE 5.11 ERRORS $\times 10^3$ (AFTER 50 TIME STEPS) $h = 0.1$, $r = 2.0$

(γ_1, γ_2) \ x	0.1	0.2	0.3	0.4	0.5
(1, 0.25)	0.206	0.321	0.306	0.234	0.199
(0.8,0.15)	0.225	0.399	0.319	0.229	0.179
(0.6, 0.05)	0.224	0.341	0.322	0.221	0.163
(0.5, 0)*	0.254	0.372	0.339	0.233	0.174
(0.4, −0.05)	0.227	0.345	0.322	0.213	0.151
(0.2 −0.15)	0.230	0.348	0.320	0.206	0.141
(0.0, −0.25)	0.232	0.350	0.318	0.200	0.137
(−0.2, −0.35)	0.234	0.350	0.316	0.196	0.128
(−0.4, −0.45)	0.235	0.351	0.315	0.192	0.123
Evans method	9.224	17.545	24.128	28.331	29.974

**Fairweather-Gourlay* method

This problem has also been solved by the direct difference scheme (5.310) using the values (1/6, 0, 1/4), (1/6, 1/144, 1/4), (0, 0, 1/4), (0, 0, 1/2) for (β, β_1, τ) with $h = .1, k = .02$ giving $r = 2.0$. The computations are carried over 50 time steps. These computations are repeated with $r = 1/\sqrt{6}$ and also using another set of values (1/6, 1/144, 1/12) for (β, β_1, τ). In this case computations are carried over 100 time steps. The error values are given in Table 5.12.

It is seen that the high accuracy scheme (5.319) produces best results. The results obtained by using the Richtmyer method and the direct method are comparable. The error values obtained by Crandall's and Todd's scheme are much higher than the error values obtained by the other schemes.

5.11 NONLINEAR PARABOLIC EQUATIONS

We now consider difference schemes for solution of the general nonlinear partial differential equation

$$\phi\left(x, t, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right) = 0 \tag{5.328}$$

TABLE 5.12 ERRORS × 10^3, $h = 0.1$

r	(β, β_1, τ)	time steps	0.1	0.2	0.3	0.4	0.5
2.0	$\left(\frac{1}{6}, 0, \frac{1}{4}\right)$	50	0.243	0.366	0.340	0.230	0.168
	$\left(\frac{1}{6}, \frac{1}{144}, \frac{1}{4}\right)$	50	0.253	0.372	0.338	0.231	0.173
	$\left(0, 0, \frac{1}{4}\right)^*$	50	0.319	0.619	0.881	1.076	0.149
	$\left(0, 0, \frac{1}{2}\right)^{**}$	50	0.432	0.834	1.178	1.426	1.518
$\frac{1}{\sqrt{6}}$	$\left(\frac{1}{6}, 0, \frac{1}{4}\right)$	100	0.045	0.050	0.015	0.016	0.021
	$\left(\frac{1}{6}, \frac{1}{144}, \frac{1}{4}\right)$	100	0.062	0.061	0.011	0.019	0.019
	$\left(0, 0, \frac{1}{4}\right)^*$	100	0.261	0.443	0.547	0.608	0.633
	$\left(0, 0, \frac{1}{2}\right)^{**}$	100	0.230	0.408	0.540	0.650	0.702
	$\left(\frac{1}{6}, \frac{1}{144}, \frac{1}{12}\right)$	100	0.025	0.015	0.007	0.004	0.006

**Todd's* method.
***Crandall's* method.

in the region $\mathcal{R} = [a \leqslant x \leqslant b] \times [0 \leqslant t \leqslant T]$ subject to certain initial and boundary conditions. We assume that the function ϕ is continuous with respect to all six of the arguments and that $\phi_{u_t} \geqslant a^* > 0$ and $\phi_{u_{xx}} \leqslant b^* < 0$.

A general two level finite difference approximation to (5.328) is given by

$$\phi\Big[x_m, ((1-\gamma_1)t_n+\gamma_1 t_{n+1}), ((1-\gamma_1)u_m^n+\gamma_1 u_m^{n+1}),$$
$$\left(\frac{u_m^{n+1}-u_m^n}{k}\right), \frac{1}{h}\mu_x\delta_x((1-\gamma_1)u_m^n+\gamma_1 u_m^{n+1}),$$
$$\frac{1}{h^2}\delta_x^2((1-\gamma_1)u_m^n+\gamma_1 u_m^{n+1})\Big] = 0 \qquad (5.329)$$

where $0 \leqslant \gamma_1 \leqslant 1$.

The truncation error arising from the approximation of (5.328) by (5.329) is obtained in the following manner. Let U_m^n denote the theoretical value of $u(x_m, t_n)$ and it will satisfy

$$\phi\Big[x_m, (t_n+\gamma_1 k), ((1-\gamma_1)U_m^n+\gamma_1 U_m^{n+1}),$$
$$\left(\frac{U_m^{n+1}-U_m^n}{k}\right), \frac{1}{h}\mu_x\delta_x((1-\gamma_1)U_m^n+\gamma_1 U_m^{n+1}),$$
$$\frac{1}{h^2}\delta_x^2((1-\gamma_1)U_m^n+\gamma_1 U_m^{n+1})\Big] = T_m^n \qquad (5.330)$$